Lesson plan

Algebraic thinking in problem solving

Contents

1. Rationale

Students usually know how to solve linear equations. However, they may have had little practice at modelling and representing relationships involving unknowns, and they often find questions involving unknowns inaccessible. The aim of this lesson is to develop students' algebraic thinking.

The lesson starts by introducing two contexts that are both horizontal in nature: building walls out of blocks and building train tracks. These contexts support students in developing an understanding of how bar models and other diagrams can be used represent **mathematical structure** (Key Principle 1) and provide a 'way in' for students, so that **all students can make progress and have some success** (Key Principle 5).

In the main task for this lesson, students match geometry questions and word questions with the same underlying mathematical structure and use bar models to represent the mathematical structure. The focus is on students being able to solve problems using algebraic thinking, rather than algebra notation.

2. GCSE curriculum

Solving equations and inequalities

A21 translate simple situations or procedures into algebraic expressions or formulae; derive an equation, … solve the equation and interpret the solution

Properties and constructions

G3 apply the properties of angles at a point, angles at a point on a straight line, … use the sum of angles in a triangle

3. Lesson objectives

- Represent contextual problems mathematically
- Use diagrams to represent mathematical structure
- Determine the value of an unknown in a problem
- Solve problems involving angles

4. Starting points

The lesson assumes that students have some understanding of angle facts. Familiarity with bar models would be advantageous.

5. Research questions

Pedagogic focus

In what ways does the teacher develop and bring the lesson to a close to support a culture where everyone believes everyone can succeed?

Maths focus

In what ways do students use representations to access the structure of mathematical problems?

6. Lesson structure

Activity	Time (min)	Description/Prompt	Materials
Introduction	5	Introduce the context of building walls out of blocks and the relationships between wall lengths for Em, Finch and Gill.	Mini whiteboards Multilink cubes (optional) Slides 2–3
Explore/ Discuss 1	10	Ask students how many blocks Em, Finch and Gill each use if 60 blocks are used altogether. Use the ‘Approaches’ handout to explore different approaches.	‘Approaches’ handout Slides 4–9
Explore/ Discuss 2	15	Introduce the context of building train tracks. Ask students to decide which bar model correctly represents the problem of finding the longest track that can be made using a total of 25 pieces, given some constraints. Then ask students to use the bar model to answer the problem.	Slides 10–13
Explore/ Discuss 3	30	Ask students what they know about the four geometry diagrams. Hold a short whole-class discussion to check that all students have a shared understanding of the angle facts related to the geometry diagrams. Then ask students to: • match word and geometry questions with the same mathematical structure • draw diagrams to represent the mathematical structure of the questions.	‘Solving problems’ handout ‘Cards’ Scissors (if cards are not already cut) Slides 14–20
Discuss 4	10	Discuss students’ work, emphasising that the same bar model can be used for different questions with the same mathematical structure.	Slides 21–24
Practice questions	10	Ask students to answer both exam questions and discuss their thinking.	‘Practice questions’ handout Slide 25
Discuss 5	10	Discuss different ways of representing a problem and how they can be used to support the solution process.	Slides 26–27

7. Teacher guidance

Introduction

Aim	To explore possible values for three unknowns that satisfy a given relationship
Materials	Mini whiteboards, multilink cubes (optional)
Slides	Slides 2–3
Time	5 minutes

This section of the lesson introduces the context of building walls out of blocks, given certain constraints. The total number of blocks is not given so there are multiple possibilities for the wall lengths. This promotes a low-threat culture, where all students can contribute and **everyone believes everyone can succeed** (Key Principle 5).

A key element of teaching for mastery is the '**concrete, pictorial, abstract**' approach in which students develop their understanding of abstract mathematical concepts and structures, starting with concrete objects. Students may benefit from using multilink cubes (or equivalent) to model the walls.

What the students might do and what you might do

Slide 2 Tell students that Em, Finch and Gill are making walls out of blocks. Finch builds a wall that is twice as long as Em's wall and Gill builds a wall that is three times as long as Em's wall. Ask students to show possible lengths for Em's, Finch's and Gill's walls on their mini whiteboards.

Tell students to hold up their mini whiteboards showing the number of blocks in each wall. If all students have identified the same three walls (for example, the simplest case: 1 (Em), 2 (Finch) and 3 (Gill)), ask a couple of different students to explain their reasoning. If there are different walls identified within the class, ask students to explain their thinking.

If necessary, check your students' understanding by asking further questions, for example:

- A wall is 4 blocks long. Whose wall could this be?
  (It could belong to Em or Finch, but not Gill)
- Suggest a wall that could be made by Em and Gill, but not Finch.
  (e.g. 3, 9, 15, 21, which are the odd multiples of 3)

Slide 3 (optional) The walls included on slide 3 can be revealed as appropriate.

- First column: 1 (Em), 2 (Finch) and 3 (Gill)
- Second column: 2 (Em), 4 (Finch) and 6 (Gill)
- Third column: 3 (Em), 6 (Finch) and 9 (Gill)

Explore/Discuss 1

Aim	To explore different ways of approaching a problem
Materials	‘Approaches’ handout
Slides	Slides 4–9
Time	10 minutes

This section of the lesson introduces an additional constraint (total number of blocks is 60) to the wall building context. Students are presented with three different approaches to determining the number of blocks Em, Finch and Gill each use. Providing worked solutions for students with can help with **determining what they already know** (Key Principle 2), as well as **building their confidence** by giving them possible starting points (Key Principle 5). Working in pairs to explore the different approaches helps to **promote a collaborative community** where students are confident to share their ways of working (Key Principle 5).

In the previous section of the lesson, some students may have used multilink cubes (or equivalent) to represent the number of blocks in Em’s, Finch’s and Gill’s walls. Using manipulatives or diagrams helps students to understand the **mathematical structure** of the situation (Key Principle 1).

What the students might do and what you might do

Slide 4 Tell students that Em builds a wall using four blocks. Ask students to answer these questions:

1. How many blocks Finch will need?
2. How many blocks will Gill need?
3. How many blocks will Em, Finch and Gill use altogether?

After a short time, ask one or more students to come to the board and use diagrams to explain their solution(s).

Slide 5 (optional) If the class found the questions on the previous slide difficult, use the animations on this slide to help to **expose the mathematical structure** of the problem (Key Principle 1).

Slide 6 Show the question given on Slide 6 and allow the students a few minutes to think about it, either individually or in pairs. Then explain that the ‘Approaches’ handout shows three different approaches to working out how many blocks Em, Finch and Gill will each use, given that there are 60 blocks. Tell students to work in pairs and give each pair a copy of the handout. Ask students to look at, and discuss the three different approaches and to answer the questions below each approach.

After a few minutes, bring the class together. Ask students which of the three approaches they found easiest to understand. Would they have used one of the approaches described, or taken a different approach? Encourage students to listen to one another as they share their thinking with the class and consider other viewpoints.

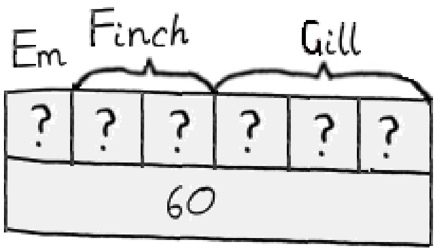
Slide 7–9 Use these slides to support a discussion of the three approaches.

Slide 7 Whilst Lara's listing approach is very methodical, it is also the least efficient. Lara could have saved time by recognising that there was a pattern, and so didn't need to fill in every cell in his table *(the number in Finch's column is 2 times the number in Em's column, the number in Gill's column is 3 times the number in Em's column, and the total number is 6 times the number in Em's column).*

Deepening understanding Ask students to describe how the numbers in Gill's column relate to the numbers in Em's and Finch's columns. Exploring this relationship helps to expose how $3x$ can be thought of both multiplicatively as $3 \times x$ and additively as $x + 2x$.

Slide 8 Meena used a letter n inside a square to represent the number of blocks used by Em. She followed this representation method through to describe the number of blocks in Finch's and Gill's walls in relation to the number of blocks in Em's wall. Using the total number of blocks as 60, Meena identified that the letter n (the 'base value') inside a square represented 10 blocks. However, she did not state the number of blocks in Finch's and Gill's walls.

Slide 9 Noah's method introduces a bar model diagram. When discussing Noah's approach, spend some time discussing the bar model first, before looking at the answers.



Many students have no experience of creating and using bar models, so you may need to spend some time unpicking exactly how Noah constructed his bar model. Make sure that students understand how Noah knew to draw 6 parts, and where the 60 came from. Emphasise that the 6 parts and the 60 end at the same place: the 6 parts make 60 altogether. *(He used 1 part to represent the number of blocks in Em's wall, 2 parts to represent the number of blocks in Finch's wall and 3 parts to represent the number of blocks in Gill's wall, resulting in a total of 6 parts. There are 60 blocks in total.)*

Ask students how Noah used his bar model to find out the number of blocks that Em, Finch and Gill each used. *(Noah divided the total of 60 blocks by the number of parts (6) to get the number of blocks in Em's walls (10). He then calculated the sizes of 2 and 3 parts to find the corresponding number of blocks in Finch's and Gill's walls.)*

When all three approaches have been discussed with the class, ask students what is the same and what is different about these methods. Draw out that all three approaches identify the base value (10) and relate the lengths of Em's and Gill's walls to this. This supports **the use of representations to expose the underlying mathematical structure**, a key element of teaching for mastery (Key Principle 1).

Explore/Discuss 2

Aim	To use diagrams to represent mathematical structure, and to use algebraic thinking to solve a problem
Materials	Mini whiteboards
Slides	Slides 10–13
Time	15 minutes

In this section of the lesson, bar models are used to support students in **identifying the mathematical structure** of a problem (Key Principle 1), and to help students solve a problem.

Algebraic notation has not been used and it is important that students recognise that the lengths of the tracks can be identified without the need to write algebraic expressions. Developing methods for solving problems involving unknown values without the need for algebra notation can help to **build students' confidence and belief that they can succeed** (Key Principle 5)

What the students might do and what you might do

Slide 10 Introduce the problem. Tell students that Em, Finch and Gill are building train tracks. They have 25 pieces of track altogether. Finch's track is twice as long as Em's. Gill's track is 5 pieces longer than Finch's. Ask students to discuss in pairs: *'What is the longest track they can each make to use all 25 pieces?'* Students do not need to answer the problem at this stage.

Slide 11 After a minute or so, tell students that Lara and Meena have tried to draw bar models to represent the relationships between the numbers of pieces in the three tracks. In their pairs, can they decide whose bar model is correct?

After a couple of minutes, ask for a show of hands for those that think Lara is correct and for those who think that Meena is correct. If there is agreement, ask a couple of students to explain their reasoning. If there is disagreement, ask pairs of students with differing views to share their thinking.

Emphasise the importance of considering problems carefully to ensure that the structure of the relationships between values is correctly identified. If students are struggling, encourage them to consider whose model correctly shows that Gill's track is 5 **pieces** longer than **Finch's** track. Confirm that Meena's model is correct, and that Lara's is not.

Slide 12 Tell students that Ollie and Pippa have also drawn bar models, but both have some mistakes in their models. Ask them to discuss these bar models in their pairs, identifying what Ollie and Pippa have done wrong.

In Ollie's bar model, the blocks making up Em's, Finch's and Gill's walls do not come to 25. The 25 bar is too long. This means that it is difficult to use the model to envisage what is left after 5 has been subtracted from 25.

Pippa has confused the base unknown value (Em's wall) with the numerical value of 5. The way she has drawn it treats each unit of the 5 as the same as Em's value, which means that Gill's wall is 7 times as long as Em's.

Slide 13 Ask students to work in pairs to use Meena's bar model to answer the question. After a few minutes, ask a couple of pairs of students to describe their approach.

Check whether students have correctly identified that ? represents 4 pieces, and check whether students have correctly identified the number of pieces in each of the three tracks (Em: 4 pieces long, Finch: 8 pieces long, Gill: 13 pieces long).

Students often lose sight of the original problem during the solution process and it is important to remind them of the need to relate any solution that they obtain back to the question posed. In this case, the length of each of the three tracks needs to be worked out and stated.

It is important that students understand how the bar model is used to solve the problem because they will depend on this understanding later in the lesson.

Explore/Discuss 3

Aim	To identify and solve problems with the same mathematical structure, using representations to support the process of determining the unknown value
Materials	'Solving problems' handout, 'Cards', scissors (if the cards have not already been cut out) and mini whiteboards
Slides	Slides 14–20
Time	30 minutes

In this section of the lesson, students match up geometry questions and word questions that have the same underlying mathematical structure, and then use diagrams to help them to find the unknown values. This lesson suggests that students draw bar models. However, allow students to use any representation that helps them to make sense of the problem (provided it accurately represents the mathematical structure).

Establishing students' existing knowledge is an important part of the mastery approach (Key Principle 2) and this section of the lesson begins with an opportunity for students to apply their knowledge of angles to four geometry diagrams. Students take turns to describe the diagrams in as much detail as they can and this **collaborative approach** provides an opportunity for students to **share their understanding** and **encourages the belief that all students have something to contribute** (Key Principle 5).

What the students might do and what you might do

Slide 14 Establish guidelines for working in pairs.

Slide 15 Distribute a copy of the 'Solving problems' handout and 'Cards' C1 to C4 to each pair of students. Ask students write down what they know about the angles in each diagram.

Students do not need to work through the diagrams in order and they may be able to say more about some of the diagrams than others. Students should be encouraged to say whatever they can about the angles in each diagram.

While the expectation at this stage is not necessarily to calculate any missing angles, students may begin to do so, as they record what they can say about each of the geometry diagrams.

Slides 16–19 After about five minutes, bring the class together. Discuss each of the geometry diagrams in turn, encouraging students to share their thinking about the angles in each one.

For example, a variety of things could be said about the diagram in row A:

- The largest angle is 3 times the size of the smallest angle.
- One angle is twice the size of another angle.
- The angles add up to 180°.
- The sum of the two smaller angles is equal to the largest angle.

Row A (slide 16): Establish that the sum of angles in a triangle is 180°

Row B (slide 17): Establish that the sum of angles on a straight line is 180°

Card C1 (slide 18): Establish that the sum of angles around a point is 360°.
Some students may incorrectly think that the $5w$ angle is 180°.

Card C2 (slide 19): Students need to recognise that the geometry diagram represents a right angle and so identify that $2z° + 10° = 90°$.

Slide 20 Explain that the task also includes four word questions with the same relationships between the unknowns as those in the geometry diagrams. Explain that this means that the same bar model can be used to represent the geometry question and the word question.

Ask students to:

- place cards C1 to C4 in the table
- draw bar models that could be used to represent the maths.

Some students may prefer to do the card matching first, and others might choose to draw the bar models first. Either way, they should take turns to fill a cell in the table, explaining their reasons to their partner.

Students may not complete all four rows of the table and it is not essential that they do so. If students are struggling, ask them to focus on rows A and B as the relationships between the unknowns in these two rows mirror those explored earlier in the lesson. Also explain that they need to find a 'base value' that all the other values relate to (one question mark in Meena's bar model, for example). It may be helpful to fold the table in half if students are finding working with all four rows overwhelming.

As you circulate around the class, observe what the students are doing, but try not to intervene, other than reminding them that, in each row, the word problem and the geometry diagram are essentially the same. Notice how different pairs approach the task and use your observations to inform the discussion in the next part of the lesson.

Discuss 4

Aim	To establish that the same representation can be used for different problems with the same mathematical structure
Slides	Slides 21–25
Time	10 minutes

The aim of this lesson is to develop students' algebraic thinking and so algebra notation may not feature at all as students complete the task. Instead, this lesson focuses on using bar models (or other diagrams) to represent the problem and to use these bar models to find the unknowns.

The bar models on slides 21–24 provide examples of possible diagrams students may produce when representing the problems. These can be used as appropriate, and it may not be necessary to reveal them all.

What the students might do and what you might do

The rows of the table do not need to be discussed in order. Instead, draw on what you observed as you circulated to decide what to discuss first. It is important to **value all students' contributions** (Key Principle 5). Encourage other students in the class to add anything that may not have already been articulated by the pair of students that are explaining their work.

Slides 21–24 For all rows of the table:

- Check that students correctly matched the word questions and the geometry questions with the same underlying mathematical structure.
- Ask students to explain how the bar models they drew relate to the two questions.
- Ask students to explain how their bar models represent the problem situation
- Emphasise the importance of relating any solutions that students have found back to the problem.

Slide 21 (row A) The relationships between the unknowns in the problems in row A are the same as between the lengths of Em's, Finch's and Gill's walls introduced at the start of the lesson. It may be helpful to refer back to the bar model used in Noah's approach (slide 9) and discuss what's the same and what's different.

Slide 22 (row B) The relationships between the unknowns in the problems in row B are the same as between the lengths of Em's, Finch's and Gill's train tracks introduced in Explore 2. Again, it may be helpful to refer back to the bar model used on slide 11 and discuss what's the same and what's different.

If students struggle to find the unknown, use the bar model to support them in understanding why you need to subtract 5 from the total, before dividing by 5.

Slide 23 (row C) Students do not need to draw a bar model for row C as one is included in the table. Some students may use this bar model to help them with the matching process, whereas others may use it during the solution process only.

Deepening understanding

- Ask students what kind of triangle is in row A. Some of them may notice that it is a right-angled triangle, without needing to calculate the values of $2y$° and $3y$°.
- Ask students how the bar model in row B could be modified to demonstrate that you need to subtract 5 from the total before dividing by 5. Some students may suggest that you could remove the 5 part, and shorten the total bar to represent 175 rather than 180.

Slide 25 Remind students that for each row the relationships between the unknowns are the same for the word question and the geometry diagram. They have the same mathematical structure and can be represented using the same bar model.

Practice questions

Aim	Students apply their knowledge to exam questions
Materials	'Practice questions' handout
Slides	Slide 26
Time	10 minutes

In this part of the lesson students apply what they have learned in the lesson to practice questions. Give students a couple of minutes to work on the questions individually and then discuss their approaches.

What the students might do and what you might do

Slide 26 Distribute a copy of the practice questions to each student and give them a couple of minutes to come up with a solution to either one or both of the problems. Make sure that students understand that these are two separate problems, not a pair of problems with the same underlying mathematical structure.

Notice whether students draw a bar model to represent the problems. If they do, do they use 360 as the total for the bars in the first question, or do they simplify the problem straight away by considering the two unknown angles adding up to 270? Once students have found a value for x, encourage them to check that $2x° + 3x° + 90° = 360°$.

Notice whether students use 'Hilary' as the base value for the second question. It is possible to draw a bar model using 'Imogen' as the base value, but students may run into problems representing the fact that Hilary has three fewer cards.

Deepening understanding The first practice question involves a $2x$ unknown and a $3x$ unknown, but the mathematical structure of the problem is different to the problems in row A of the task / Em's, Finch's and Gill's walls in the introductory activities (which also involve a $2x$ and a $3x$). Can students explain why this is? Do students recognise that the base value (x) is calculated but does not actually feature in the bar model?

Discuss 5

Aim	To explore different ways of representing the problem
Materials	Mini whiteboards
Slides	Slides 27–28
Time	10 minutes

It is important to **value and make sense of students' different ways of working** to increase students' self-belief (Key Principle 5). In this final part of the lesson, discuss not only the answers to the questions, but also how students approached them.

What the students might do and what you might do

Slide 27–28 Begin by asking students how they answered the questions. It is important that you spend a bit of time on what they did as well as what the correct answers are. Ask students that drew a bar model to describe their thinking and explain in what ways their bar model helped them to make sense of the problem.

Show students possible bar models to represent the practice questions and discuss as a class the questions related to these possible bar models. For the first question, three different bar models are given, and students should understand that although they are different, they are all correct. Make sure that they know how to use the bar models to find the value of x.

Meena's simplifies the first problem by subtracting the right angle from the total, so that angle sum is 270° rather than 360°. Both Lara's and Meena's bar models make the need to divide by 5 clear and Noah's bar model makes the unknown value of x explicit.

For the second question, only Noah's bar model is provided. His first bar model mixes up numbers and the unknown (Hilary's number of cards), and he then simplifies it by putting all the unknown's together and adding up all the numbers. This means that to find H, he has to subtract one number (9) from the total.